LAMINAR FLOW PAST A STEP

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Boundary-layer separation is caused by many factors: adverse pressure gradient, shock impingement on the boundary layer, discontinuity in the surface contour, etc. One of the examples of a body with surface discontinuity is a step on a flat plate. Such a configuration is often present in practical situations and a large number of experimental studies have been conducted [1] on flow past a step. Numerical modeling of flow past a step using full Navier–Stokes equations have been carried out for a limited range of Reynolds numbers (see, e.g., [2]).

The method of matched asymptotic expansions has been widely used to investigate disturbed flow in a boundary layer at high Reynolds numbers. A survey of literature on the application of this method to the analysis of separated flows can be found in [3-5]. One of the important factors in the application of matched asymptotic expansions is the delineation of characteristic regions of the flow. The division of the flow into different regions is associated with nonuniform applicability of asymptotic expansions or with different effects of convection, diffusion, pressure gradient, and others on the flow. The objective of the present work is the investigation of the structure of disturbed flow past a step for a range of parameters proportional to the height of the step. One of the characteristic regions of such a flow is the region with length scales equal to the order of the step height. In another characteristic region the disturbance is propagated upstream from the step. Analysis of these regions is presented below.

1. Consider a laminar flow past a step located at a distance ℓ from the leading edge of a flat surface. The origin of the cartesian coordinate system coincides with the plate leading-edge (Fig. 1). The distances along the plate and normal to it and the corresponding velocity components, density, pressure, and dynamic viscosity coefficient are denoted by: $x\ell$, $y\ell$, $u_\infty u$, $u_\infty v$, $\rho_\infty\rho$, $\rho_\infty u^2_\infty p$, $\mu_\infty\mu$. It is assumed that $\mathrm{Re} = \rho_\infty u_\infty \ell/\mu_\infty = \varepsilon^{-2}$($\rho_\infty$, u_∞, μ_∞ are the density, velocity, and dynamic viscosity coefficient in the undisturbed flow) is large, but it does not exceed the critical value when transition occurs upstream of the step. It is also assumed that the step height H can vary within the range $\varepsilon^{3/2} \leqslant H \leqslant \varepsilon^{5/4}$. As shown in [6], when $H \sim O(\varepsilon^{3/2})$, the disturbed flow near the step in the region with length scales $x \sim y \sim H$ is described by the complete system of Navier–Stokes equations for incompressible flows. The effect of viscosity in this region becomes negligible with increase in step height. To the first approximation, the flow near the step with $H > \varepsilon^{3/2}$ is described by the system of Euler equations for incompressible flow. The following expressions are introduced for the coordinates and flow variables in the region 1 near the step (see Fig. 1):

$$x = 1 + Hx_1,\ y = Hy_1,\ u = H\varepsilon^{-1}au_1(x_1, y_1) + \ldots,\ v = \\ = H\varepsilon^{-1}av_1(x_1, y_1) + \ldots,\ \rho = \rho_w + \ldots,\ p = p_\infty/\rho_\infty u_\infty^2 + \\ + H^2\varepsilon^{-2}a^2\rho_w p_1(x_1, y_1) + \ldots,\ a = \varepsilon\partial u/\partial y\,(1, 0). \quad (1.1)$$

Substituting (1.1) in Navier–Stokes equations and letting $\varepsilon \to 0$ and $H \to 0$ lead to

$$u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1} + \frac{\partial p_1}{\partial x_1} = 0,\ u_1\frac{\partial v_1}{\partial x_1} + u_1\frac{\partial v_1}{\partial y_1} + \frac{\partial p_1}{\partial y_1} = 0,\ \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} = 0. \quad (1.2)$$

The no-slip boundary conditions specified on the step and the flat plate are

$$u_1(0,\ 0 < y_1 < 1) = 0,\ v_1(x_1 \leqslant 0,\ 0) = v_1(x_1 > 0.1) = 0. \quad (1.3)$$

The boundary conditions for the system (1.2) specified as $x_1 \to -\infty$ depend on the nature of the upstream influence of the step. It is shown below that nonlinear variation in velocity profile in the region of a height comparable with the step height and a length exceeding the step height are caused by steps whose heights are an order of magnitude greater than $O(\varepsilon^{5/4})$. When $H < O(\varepsilon^{5/4})$ and as $x_1 \to -\infty$, the boundary conditions take the form

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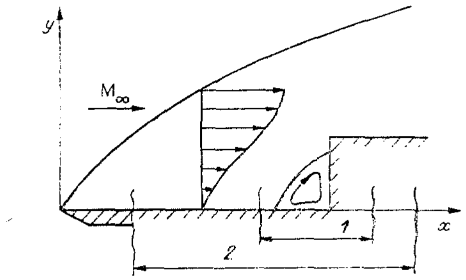

Fig. 1

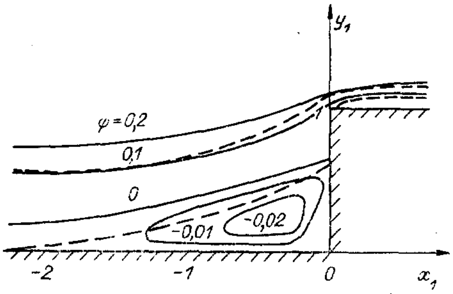

Fig. 2

$$u_1(x_1 \to -\infty,\ y_1) = y_1. \tag{1.4}$$

The boundary conditions are similar at a far downstream location:

$$u_1(x_1 \to \infty,\ y_1) = y_1. \tag{1.5}$$

The condition for damping at large distances normal to the surface leads to

$$u_1(x_1,\ y_1 \to \infty) = y_1 + O(1). \tag{1.6}$$

The solution to the boundary-value problem (1.2)-(1.6) is nonunique. Recirculation zones are formed ahead of the step where viscosity plays a crucial role. Thus, if the vorticity is constant in the recirculation region, then it is possible to determine its value from the condition of spatial periodicity of the flow in the boundary layer located at the edge of the recirculation region [7]. As an example of a possible solution to the boundary-value problem (1.2)-(1.6), it has been assumed in the present paper that the vorticity in the recirculation region coincides with the vorticity in the undisturbed wall layer upstream of the step. A sketch of streamlines obtained by using complex variables is shown in Fig. 2, where the dashed lines are streamlines obtained from computations based on Navier-Stokes equations using the method described in [6]. In these computations, $Re_1 = \rho_w a H^2/\mu_w \varepsilon^3$ was equal to 10^4. A qualitative agreement is observed in streamlines obtained from the solution of Euler and Navier-Stokes equations. A step of height $H \sim O(\varepsilon^{3/4})$ leads to a change in velocity profile in the wall layer ahead of the step with a thickness comparable to the height of the step. For further analysis it is necessary to obtain a solution to the Euler equations $u_1(x_1 \to \infty, y_1)$ with arbitrary initial velocity profile $u_1(x_1 \to -\infty, y_1)$. In order to find the desired solution, it is possible to use the conservation of total pressure along a streamline. Let $f_0 = u_1(x_1 \to -\infty, y_1)$ and $f_1 = u_1(x_1 \to \infty, y_1)$; then from Bernoulli's equation

$$f_1(\psi) = \sqrt{f_0^2(\psi) - 2\Delta p}, \tag{1.7}$$

where Δp is the pressure drop induced by the step $[\Delta p = p_1(x_1 \to \infty) - p_1(x_1 \to -\infty)]$; ψ is the stream function determined from

$$f_0 = d\psi/dy_1. \tag{1.8}$$

The equations (1.7) and (1.8) give the shape of the veocity profile $f_1(\psi)$. In order to determine the unknown pressure drop Δp, it is necessary to use the boundary condition (1.6), which has the form

$$\lim_{\psi\to\infty}\left[1 + \int_0^\psi \frac{d\psi}{f_1} - 2\psi\right] = \lim_{\psi\to\infty}\left[\int_0^\psi \frac{d\psi}{f_0} - 2\psi\right]. \tag{1.9}$$

The pressure drop Δp can be determined from Eq. (1.9) by numerical integration using an iterative procedure. The condition (1.9) is equivalent to the condition of conservation of total displacement thickness, comprised of the variable part of the displacement thickness and the step height. The variation in the total displacement thickness over short distances could lead to the appearance of large pressure drops that exceed, by an order of magnitude, the dynamic pressure present at transverse locations which are at distances of the order of the step height normal to the surface. In turn, such changes in velocity would cause even larger changes in the total displacement thickness, i.e., they may lead to a non-self-consistent flow situation discussed in detail in [8]. The variation in displacement thickness takes place over a more extensive region, the so-called free-interaction region [4, 5].

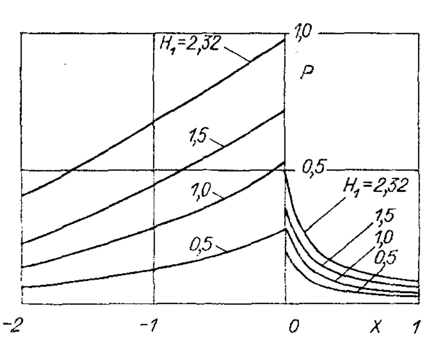

Fig. 3

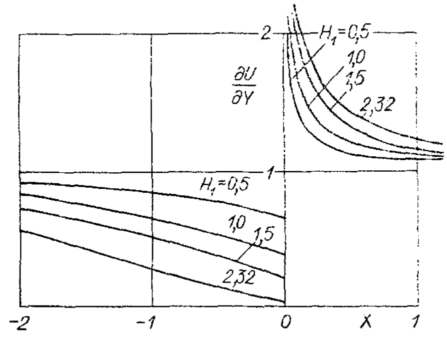

Fig. 4

2. The above described estimates of the pressure drop are obtained in the free-interaction region 2

$$\Delta p \sim H^2/\varepsilon^2, \tag{2.1}$$

from which it follows that as $H \sim \varepsilon^{5/4}$, the pressure drop $\Delta p \sim \varepsilon^{1/2}$ is equal, by an order of magnitude, to that pressure drop for which nonlinear variation in shear stresses is induced in the region 2 of length $\Delta x \sim \varepsilon^{3/4}$ ahead of the step [4, 5]. All these are applicable to subsonic as well as supersonic external flows. Disturbances propagating upstream from the step lead to a change in displacement thickness consistent with pressure fluctuations. Near the step, in the region 1, a shocklike jump (of the scale of the free-interaction region 2) in pressure takes place for a constant displacement thickness. The streamwise velocity profile changes according to (1.7). This is followed further downstream by a damping of disturbances induced by the step in the free-interaction region ($\Delta x \sim \varepsilon^{3/4}$). The boundary-value problem describing the flow in the free-interaction region at supersonic speeds has the form [4, 5]

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + \frac{\partial P}{\partial X} = \frac{\partial^2 U}{\partial Y^2}, \tag{2.2}$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0,\quad \frac{\partial P}{\partial Y} = 0,\quad P = -\frac{\partial}{\partial X}\lim_{Y\to\infty}(U - Y),$$

$$U(X\to -\infty, Y) = Y,\quad \partial U/\partial Y(X, Y\to\infty) = 1,$$

$$V(X \leqslant 0, 0) = 0,\quad V(X > 0, H_1) = 0,$$

$$U(X\to\infty, Y) = Y - H_1,\quad U(X\leqslant 0, 0) = 0,\quad U(X > 0, H_1) = 0,$$

$$U = \varepsilon^{-1/4}\mu_w^{-1/4}a^{-1/4}\rho_w^{1/2}\beta^{1/4}u,\quad V = \varepsilon^{-3/4}\mu_w^{-3/4}a^{-3/4}\rho_w^{1/2}\beta^{-1/4}v,$$

$$P = \varepsilon^{-1/2}\mu_w^{-1/2}a^{-1/2}\beta^{1/2}p,\quad X = \varepsilon^{-3/4}a^{5/4}\mu_w^{1/4}\rho_w^{1/2}\beta^{3/4}x,$$

$$Y = \varepsilon^{-5/4}\mu_w^{-1/4}a^{3/4}\rho_w^{1/2}\beta^{1/4}y,\quad H_1 = \varepsilon^{-5/4}\mu_w^{-1/4}a^{3/4}\rho_w^{1/2}\beta^{1/4}H.$$

The boundary-value problem (2.2) should be supplemented by Eqs. (1.7)-(1.9) coupling velocity profiles at $X \to -0$ and $X \to +0$, and also the pressure fluctuation to the left and right of the step face. From these relations it follows, in particular, that the streamwise velocity on the upper surface of the step becomes nonzero (zero streamwise velocity along the zero streamline under the action of the pressure drop ΔP becomes equal to $u_w = \sqrt{-2\Delta P}$). We observe that $\Delta P < 0$, since, for the conservation of total displacement thickness, the incompressible flow near the step should accelerate.

Consider first the solution to the problem (2.2) for the small parameter $H_1 \ll 1$ or for $H < O(\varepsilon^{5/4})$. The problem (2.2) can then be linearized:

$$U = Y + U_1H_1 + \ldots,\quad V = V_1H_1 + \ldots,\quad P = P_1H_1 + \ldots, \tag{2.3}$$

$$Y\frac{\partial U_1}{\partial X} + V_1 + \frac{\partial P_1}{\partial X} = \frac{\partial^2 U_1}{\partial Y^2},\quad \frac{\partial U_1}{\partial X} + \frac{\partial V_1}{\partial Y} = 0,\quad P_1 = -\lim_{Y\to\infty}\frac{\partial U_1}{\partial X},$$

$$U_1(X\to -\infty, Y) = 0,\quad \partial U_1/\partial Y(X, Y\to\infty) = 0,\quad U_1(X\to\infty, Y) = -1,$$

$$U_1(X < 0, 0) = 0,\quad U_1(X > 0, 0) = -1,\quad V_1(X, 0) = 0.$$

We note that the pressure drop in the region 1 for $H_1 \ll 1$ is of higher order ($\Delta P_1 = O(H_1^2)$). The solution to the linear problem (2.3) is obtained in [9] for the flow in near the starting and end points of the motion of the surface.

For finite values of the parameter H_1, the solution to the problem (2.2) may be obtained numerically. Here, it is solved using the numerical method of [9]. The principal difficulty in the integration of Eq. (2.2) is associated with the appearance of nonzero velocity at the surface when $X > 0$. A subregion is introduced within the computational domain to overcome this problem and obtain a correct solution. The need for the introduction of such a subregion is associated with the formation of a boundary layer at the upper surface of the step.

The solution to the problem (2.2) is obtained for a number of values of the parameter H_1. The distribution of pressure fluctuations on the plane ($X < 0$) and on the step surface ($X > 0$) are shown in Fig. 3. The pressure fluctuation upstream of the step also increases with increase in the step height H_1. For the range of values of the parameter H_1 studied here, there is a characteristic monotonic increase in the pressure drop ΔP with increase in H_1. The solution, expressed as a power series $X(P = P(0) + C_1X^{1/2})$ for $X \to +0$, was obtained in [9]. The second term in this series represents the accelerating flow required to conserve the local total displacement thickness with increase in the developing boundary-layer thickness. The numerical procedure for the integration of the boundary-value problem (2.2) was the following: At a certain point upstream of the step, a positive pressure fluctuation was specified, and the problem was subsequently solved by marching technique. The choice of the initial pressure disturbance was made from the condition for the damping of disturbances at large distances downstream of the step. When the initial disturbance is too small, the solution ended at the singluar point X_0, where $P(X \to X_0) \to -\infty$. Such a solution describes a flow near the step of finite length for which a large negative pressure drop is specified at the base section (at $X = X_0$). The solutions corresponding to too large an initial pressure disturbance were characterized by an increase in pressure in the region downstream of the step surface, which finally resulted in a decrease in skin-friction to zero. A solution of this type corresponds to the flow past a step with a flap installed at a certain distance from the step surface, which causes flow separation.

The skin-friction distribution along the flat plate ($X < 0$) and along the step surface ($X > 0$) is shown in Fig. 4. There is a decrease in skin-friction upstream of the step with increase in step height. When $X = 0$, there is a shock-like jump in the value of skin friction to infinitely large values, which is associated with the formation of a boundary layer along the step surface. Downstream of the step face, for all values of the parameter H_1 under investigation, the surface skin-friction monotonically decreases to the value of skin-friction in the undisturbed boundary layer.

The solution to the problem (2.2) was obtained for a finite range of variation of the parameter H_1, corresponding to attached (pre-separated) flow upstream of the step. For the computation of the flow with $H_1 \geqslant 2.5$, the finite-difference scheme should permit the presence of a reverse-flow region upstream of the step. Principles of setting up the computational scheme for such a flow situation are given in [9]. It is worth mentioning that such a reverse-flow region always exists in the flow upstream of the step, and we are concerned about the possible increase in the dimensions of such a region by a scale comparable to the length of the free-interaction region. An increase in the step height will lead to an increase in the dimensions of the separation region and to an upstream displacement of the separation point.

LITERATURE CITED

1. P. Chen, Flow Separation, Vol. 1-3, Mir, Moscow (1973).
2. V. I. Myshenkov, "Numerical solution of Navier—Stokes equations for the flow past a delta wing," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 4 (1972).
3. V. V. Sychev and A. I. Ruban, "Asymptotic theory of laminar boundary-layer separation of incompressible flows," Usp. Mekh., 2, No. 4 (1979).
4. V. Ya. Neiland, "Asymptotic theory of flow separation and boundary-layer interaction with supersonic flow," Usp. Mekh., 4, No. 2 (1981).
5. K. Stewartson, "Some recent studies in triple deck," in: Numerical and Physical Aspects of Aerodynamic Flows, No. 4, Springer-Verlag, New York—Berlin (1982).
6. V. V. Bogolepov, "Computation of flow past a forward facing step," Zh. Prikl. Mekh. Tekh. Fiz., No. 2 (1983).
7. V. Ya. Neiland and V. V. Sychev, "Theory of flows in steady separated regions," Uchen. Zap. Tsent. Aéro-Gidro. Inst., 1, No. 1 (1970).
8. V. V. Bogolepov and V. Ya. Neiland, "Flow past small protuberances on a surface in supersonic viscous flow," Tr. Tsent. Aéro-Gidro. Inst., No. 1363 (1971).
9. I. I. Lipatov and V. Ya. Neiland, "Effect of instantaneous change in the motion of a flat plate surface in laminar; supersonic boundary layer," Uchebn. Zap. Aéro-Gidro. Inst., 13, No. 5 (1982).